JOURNAL OF MATERIALS SCIENCE 19 (1984) 159–166

Application of fracture mechanics to heterogeneous systems – prediction of fatigue life of ceramics

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A prediction of fatigue life and fracture stress of ceramics of a heterogeneous system was made by application of fracture mechanics based on slow crack growth. It was shown that the prediction of fatigue life and fracture stress in ceramics of heterogeneous system in general could be dealt with in a similar way to those of ceramics of a homogeneous system. Experimental examination of the validity of the prediction was made with LAS glass ceramics (Li_2O–Al_2O_3–SiO_2) at various stages of devitrification as well as foamed glass. These results proved the validity of the formulae derived.

1. Introduction

Many works on the prediction of fatigue life in ceramics have been made successfully on the basis of the subcritical crack growth, and many useful formulae are presented. These works appear to be limited to ceramics of a homogeneous system. A substantial number of ceramics, however, belong to heterogeneous systems composed of crystals and glasses, crystals and voids, glasses and voids, and many kinds of crystals. Then, the study of fatigue life of ceramics with a heterogeneous system on that basis is very important.

The aim of the present paper is to study the fatigue life and fracture stress of ceramics of a heterogeneous system on the basis of the subcritical crack growth.

2. Theory

In ceramics of a heterogeneous system, each phase constructing the ceramic is randomly dispersed in the form of a particle, and a crack propagates through the phases.

As the propagation speed is different from one phase to another, the crack front does not always follow a straight line. The averaged propagation speed, however, is thought to be nearly the same, and the crack front can be assumed to be straight. The speed or the fatigue life will be given by a suitable model in which each phase, in the form of a particle with an averaged size, is dispersed in proportion to the concentration of each phase. As a model, a slab form, in which two phases are dispersed regularly, will be dealt with for simplicity. The model is shown in Fig. 1. In the model, the phase, A, makes a particle of the size of δ_A, and the phase, B, makes that of δ_B. A crack with the length of a_i is assumed to be located initially in the phase, A, whose depth from the surface is ϵ.

The crack propagation in ceramics of single phase is described by the following equation [1];

$$v = \frac{da}{dt} = AK_I^n = AY^n\sigma^n a^{-n/2} \qquad (1)$$

where v, a and t stand, respectively, for crack propagation speed, crack length and time, and A, K_I, Y, σ and n stand, respectively, for a constant depending on humidity and temperature, stress intensity factor, geometrical constant, stress and material constant. In most ceramics, Equation 1 can well be approximated by the following equation [2]:

$$a_i^{(2-n)/2} - a^{(2-n)/2} = \left(\frac{n-2}{2}\right) A\sigma_0^n \int_0^t Y^n f^n(t)\,dt$$
$$\sigma = \sigma_0 f(t) \qquad (2)$$

where σ_0 stand for an arbitrarily chosen representative stress and $f(t)$ is a function of time. In most cases, Y is independent of time, and it can be taken out of the integration. Moreover, in many cases, σ is a periodic function, and σ_0 is conveniently represented by the maximum stress, σ_m, in the period.

Equation 2 holds inside each particle, and the fatigue life as well as the fracture stress is given by the successive application of the equation to each particle. On the assumption that the crack propagates from one particle into another without meeting any obstacle, the fatigue life, L_f, is given by summing up the time needed for the crack to pass through each particle, t_1, t_2, t_3 and so on, where t_j represents the time to pass through the j-th particle, counted from the initial location of the crack.

$$L_f = \sum_{j=1}^{\infty} t_j \tag{3}$$

$$\left.\begin{aligned}
& a_i^{(2-n_A)/2} - (\epsilon + \delta_A)^{(2-n_A)/2} \\
& \quad = H_A \sigma_m^{n_A} \int_0^{t_1} f^n \, dt \\
& (\epsilon + k\Delta)^{(2-n_A)/2} - (\epsilon + k\Delta + \delta_A)^{(2-n_A)/2} \\
& \quad = H_A \sigma_m^{n_A} \int_{\bar{t}_{2k}}^{\bar{t}_{2k+1}} f^n \, dt \\
& (\epsilon + k\Delta - \delta_B)^{(2-n_B)/2} - (\epsilon + k\Delta)^{(2-n_B)/2} \\
& \quad = H_B \sigma_m^{n_B} \int_{\bar{t}_{2k-1}}^{\bar{t}_{2k}} f^n \, dt \\
& \bar{t}_k = \sum_{j=1}^{k} t_j \\
& H_A = \left(\frac{n_A - 2}{2}\right) A_A Y^{n_A} \\
& H_B = \left(\frac{n_B - 2}{2}\right) A_B Y^{n_B} \\
& \Delta = \delta_A + \delta_B \\
& \sigma = \sigma_m f(t) \quad (|f(t)| \leqslant 1) \\
& k = 1, 2, 3 \ldots
\end{aligned}\right\} \tag{4}$$

where the suffixes, A and B, indicate that the variables with the suffix represent those referring to the phase, A or B.

For a periodically changing stress, the initial and final time of the integration in Equation 4 does not always agree with the beginning and the end of the period. In practice, however, as the fatigue life is much longer than the period, the approximation of shifting the initial time of the integration to the beginning of the period and the final one to the end affects the result little. Therefore, Equation 4 can be written as follows:

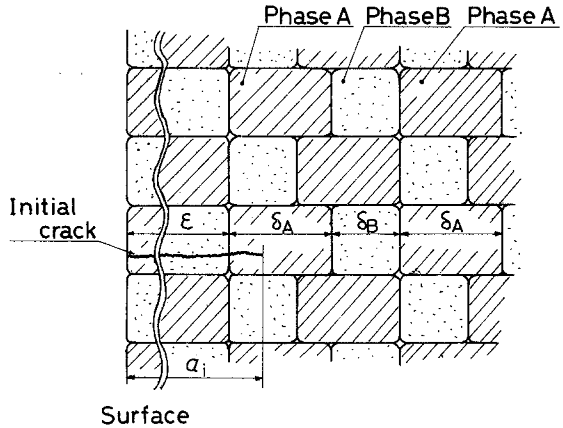

Figure 1 Geometry of slab model showing microstructure for ceramics of a heterogeneous system.

$$\left.\begin{aligned}
& a_i^{(2-n_A)/2} - (\epsilon + \delta_A)^{(2-n_A)/2} \\
& \quad = H_A \sigma_m^{n_A} \int_0^{t_1} f^n \, dt \\
& (\epsilon + k\Delta)^{(2-n_A)/2} - (\epsilon + k\Delta + \delta_A)^{(2-n_A)/2} \\
& \quad = H_A \sigma_m^{n_A} N_{2k+1} \tau_{\mathrm{eff}} \\
& (\epsilon + k\Delta - \delta_B)^{(2-n_B)/2} - (\epsilon + k\Delta)^{(2-n_B)/2} \\
& \quad = H_B \sigma_m^{n_B} N_{2k} \tau_{\mathrm{eff}} \\
& \tau_{\mathrm{eff}} = \int_0^{\omega} f^n \, dt
\end{aligned}\right\} \tag{5}$$

$$t_j \doteqdot N_j \omega \tag{6}$$

where N_{2k+1}, N_{2k} and ω stand, respectively, for the number of the repeating periods needed to pass through the $(2k + 1)$-th particle, that of the $2k$-th one, and the periodic time.

For static stress, Equation 4 gives the following equations:

$$\left.\begin{aligned}
& a_i^{(2-n_A)/2} - (\epsilon + \delta_A)^{(2-n_A)} \\
& \quad = H_A \sigma_m^{n_A} t_{1,s} \\
& (\epsilon + k\Delta)^{(2-n_A)/2} - (\epsilon + k\Delta + \delta_A)^{(2-n_A)/2} \\
& \quad = H_A \sigma_m^{n_A} t_{2k+1,s} \\
& (\epsilon + k\Delta - \delta_B)^{(2-n_B)/2} - (\epsilon + k\Delta)^{(2-n_B)/2} \\
& \quad = H_B \sigma_m^{n_B} t_{2k,s}
\end{aligned}\right\} \tag{7}$$

where the suffix s, indicates that the variable represents that under static stress.

From the comparison of Equation 5 with Equation 7, the following equation is derived for the fatigue life of ceramics under the static stress of σ_m and that under periodically changing stress of the common maximum value of σ_m,

$$t_{j,s} \doteqdot N_j \tau_{eff} \quad (8)$$

From Equations 3, 6 and 8, the following equations can be given, for the ceramics in which $H_A^{-1}\sigma_m^{-n_A}\delta_A \gg$ or $\ll H_B^{-1}\sigma_m^{-n_B}\delta_B$, or in which $n_A \simeq n_B$

$$L_{f,c} = k_c L_{f,s} \quad (9)$$

$$k_c = \omega/\tau_{eff} \quad (10)$$

where $L_{f,c}$ and $L_{f,s}$ represent, respectively, the fatigue life of a ceramic under a periodically changing stress of the maximum value of σ_m and that of the same ceramic under the static stress of σ_m. The relation is substantially the same as the formulae connecting the static fatigue life and cyclic one, derived by Evans and Fuller [3] for a ceramic of homogeneous system.

Equation 9 shows that the cyclic fatigue life of ceramics can be known from the static fatigue life of the ceramics in which one phase contributes mainly to the life, or $H_A^{-1}\sigma_m^{-n_A}\delta_A \gg$ or $\ll H_B^{-1}\sigma_m^{-n_B}\delta_B$, or in which $n_A = n_B$, in a heterogeneous system, too. But, Equation 9 does not always hold in a heterogeneous system, and the cyclic fatigue life can not be given only by multiplying some constant to the fatigue life. This is one of the features of a heterogeneous system.

The fracture stress, σ_f, can be given by substituting L_f for the time in $f(t)$, or

$$\sigma_f = \sigma_m f(L_f) \quad (11)$$

One of the most important applications of the equation is that for static fatigue life, as well as for fracture stress which occurs under a linearly increasing stress. Because the former is the basis of the prediction of the fatigue life, including a cyclic one, and the latter is frequently a measure of mechanical properties of a ceramic.

$L_{f,s}$ can be given from Equations 3 and 7. Especially, for the ceramics of small grains (δ_A, $\delta_B \ll \epsilon$), $L_{f,s}$ can be approximated as follows:

$$L_{f,s} \doteqdot \frac{1}{H_A}\sigma_m^{-n_A}\left[a_i^{(2-n_A)/2} - (\epsilon+\delta_A)^{(2-n_A)/2}\right] + \frac{1}{H_A}\sigma_m^{-n_A}\left(\frac{\delta_A}{\delta_A+\delta_B}\right)(\epsilon+\delta_A+\delta_B)^{(2-n_A)/2} + \frac{1}{H_B}\sigma_m^{-n_B}\left(\frac{\delta_B}{\delta_A+\delta_B}\right)(\epsilon+\delta_A+\delta_B)^{(2-n_B)/2}. \quad (12)$$

For the derivation of Equation 12, the summation $\Sigma_{k=1}^{j}(\alpha+k\beta)^c$ was approximated by $(\alpha+\beta)^c + \int_1^j(\alpha+x\beta)^c\,dx$.

Equation 12 is useful to study the dependence of the fatigue life on the grain size. As seen in the equation, the fatigue life is much affected by grain size as well as the phase in which the initial crack is located. The phase is expected to be altered with the increase or the decrease of grain size. Therefore, the dependence of the fatigue life on grain size can not be expressed in a simple way, except for some special cases.

Moreover, in Equation 12, when $a_i \sim \epsilon + \delta_A$, and $H_A^{-1}\sigma_m^{-n_A}\delta_A \ll H_B^{-1}\sigma_m^{-n_B}\delta_B$, $L_{f,s}$ can be approximated as follows:

$$L_{f,s} \doteqdot H_B^{-1}\sigma_m^{-n_B}\left(\frac{\delta_B}{\delta_A+\delta_B}\right)(\epsilon+\delta_A+\delta_B)^{(2-n_B)/2} \doteqdot H_B^{-1}\sigma_m^{-n_B}\left(\frac{\delta_B}{\delta_A+\delta_B}\right)(a_i+\delta_B)^{(2-n_B)/2}. \quad (13)$$

These approximated formulae, i.e. Equations 12 and 13, hold well for a ceramic which is composed of fine particles ($\epsilon \gg \delta_A$, δ_B) as well as composed of one major phase and other minor phases ($\delta_B = 0$).

In a foamed glass of low density, the pore size (δ_A) is much larger than the wall thickness (δ_B), then Equation 13 is rewritten as follows:

$$L_{f,s} \doteqdot H^{-1}\sigma_m^{-n}\left(\frac{\delta_B}{\delta_A}\right)a_i^{(2-n)/2}. \quad (14)$$

In these approximated formulae, the fatigue life is proportional to σ_m^{-n}, which agrees well with the conclusion about the influence of the maximum stress on the life of a ceramic in a homogeneous system. Then, SPT (strength–probability–time) diagram [4] as well as T-SPT (thermal shock severity–probability–time) diagram [2] hold for such ceramics:

$$\ln(-\ln P) = \frac{m}{n}\ln N + m\ln\sigma + C_1 \text{ (SPT)} \quad (15)$$

$$\ln(-\ln P) = \frac{m}{n}\ln N + m\ln\Delta T + C_2 \text{ (T-SPT)} \quad (16)$$

where P is the cumulative survival probability for the fatigue life, N, under the mechanical stress, σ, or the thermal shock with the severity of ΔT and m and C_1 and C_2 stand, respectively, for Weibull modulus and some constant depending on material, heating process and so on. But in exactness, the life is not always proportional to the power of σ_m^{-1}, mainly due to the difference in n_A and n_B. Moreover, in general, $L_{f,s}$ is not directly proportional to $a_i^{(2-n)/2}$. Therefore, the SPT diagram as well as the T-SPT diagram cannot be composed of parallel lines in a heterogeneous system in general, even if the initial crack distribution follows the modified Weibull function ($p = \exp[-V(a_i/a_0)^{-m/2}]$) [2, 5], which is responsible for the Weibull distribution in the fracture stress of ceramics in a homogeneous system. On these points, a heterogeneous system differs from a homogeneous one.

In addition, $L_{f,s}$ is dominated by the term relating to the phase, A, and that relating to the phase, B, which is shown in Equation 12. The contribution of each term to $L_{f,s}$ is influenced by the stress in a way different from A to B and the influence in A is different from that in B, because the crack propagation speed in each phase depends on stress in different ways. Then, strictly speaking, the dependence of $L_{f,s}$ on σ_m will vary with the stress, for instance, from $\sigma_m^{-n_A}$ to $\sigma_m^{-n_B}$ and vice versa. It implies that the inclination of curves or lines in SPT and T-SPT diagrams tends to vary with stress or thermal shock severity applied to a ceramic. This variation in the inclination is thought to be a feature in a heterogeneous system in contrast to a homogeneous one in that the inclination is constant.

In a similar way, the fracture stress of a ceramic of a heterogeneous system is given in a simple analytical form, when the speed of crack propagation is much larger in one phase than in the other, or when the concentration of one phase is much larger than the other. Thus, when $\epsilon + \delta_A \gg \delta_B$ or $\epsilon + \delta_B \gg \delta_A$, the fracture stress, σ_f, is given by the following,

$$\left.\begin{aligned}
\sigma_f \doteqdot\ & H_A^{1/(n_A+1)}\dot{\sigma}^{1/(n_A+1)}\Bigg[a_i^{(2-n_A)/2} + \left(\frac{\delta_A}{\delta_A+\delta_B}\right)(\epsilon+\delta_A+\delta_B)^{(2-n_A)/2} - (\epsilon+\delta_A)^{(2-n_A)/2}\Bigg]^{1/(n_A+1)} \\
& (\epsilon+\delta_B \gg \delta_A) \\
\sigma_f \doteqdot\ & H_B^{1/(n_B+1)}\dot{\sigma}^{1/(n_B+1)}\left(\frac{\delta_B}{\delta_A+\delta_B}\right)^{1/(n_B+1)} \times (a_i+\delta_B)^{(2-n_B)/2(n_B+1)} \\
& (\epsilon+\delta_A \gg \delta_B,\ a_i \sim \epsilon+\delta_A).
\end{aligned}\right\} \tag{17}$$

In any case, the fracture stress is proportional to $\dot{\sigma}^{1/(n+1)}$ in the approximations, and the manner of dependence of the stress on $\dot{\sigma}$ is the same with that in a homogeneous system.

Equation 17 can be applied to the fracture of a ceramic in which one phase is present as a majority, or the crack propagation speed is very low in one phase, or many voids present.

3. Experimental details

To examine the validity of the formulae, some experiments on glass ceramics and foamed glass were made. The glass ceramics (Devitron) and foamed glass (Celoam) were supplied by Ishizuka Glass Co., Ltd. and Toyoda Weaving and Spinning Co., Ltd. Their compositions and typical properties are listed in Table I.

Their materials were submitted to thermal and cyclic fatigue life measurements, and fracture stress measurement.

The dimensions of the glass ceramics specimen are $4.3\phi \times 70\,mm^2$. The specimen was received as a glass rod, and before the measurement it was heat treated for devitrification. Those of the foamed glass are 21 mm x 21 mm x 100 mm. The

TABLE I Composition and typical properties of specimens

Properties	Material	
	Glass ceramics (Devitron)	Foamed glass (Celoam)
Composition	Li_2O, Al_2O_3, SiO_2	SiO_2, Na_2O, MgO
Crystalline phase	β-Spodumen	None
Density ($g\,cm^{-3}$)	2.49	0.195
Thermal expansion coefficient (° C)	2.98×10^{-6}	8×10^{-6}
Thermal conductivity ($cal\,cm^{-1}\,sec^{-1}\,°\,C^{-1}$)	4.7×10^{-3}	1.4×10^{-4}
Young's modulus ($kg\,cm^{-2}$)	8×10^5	

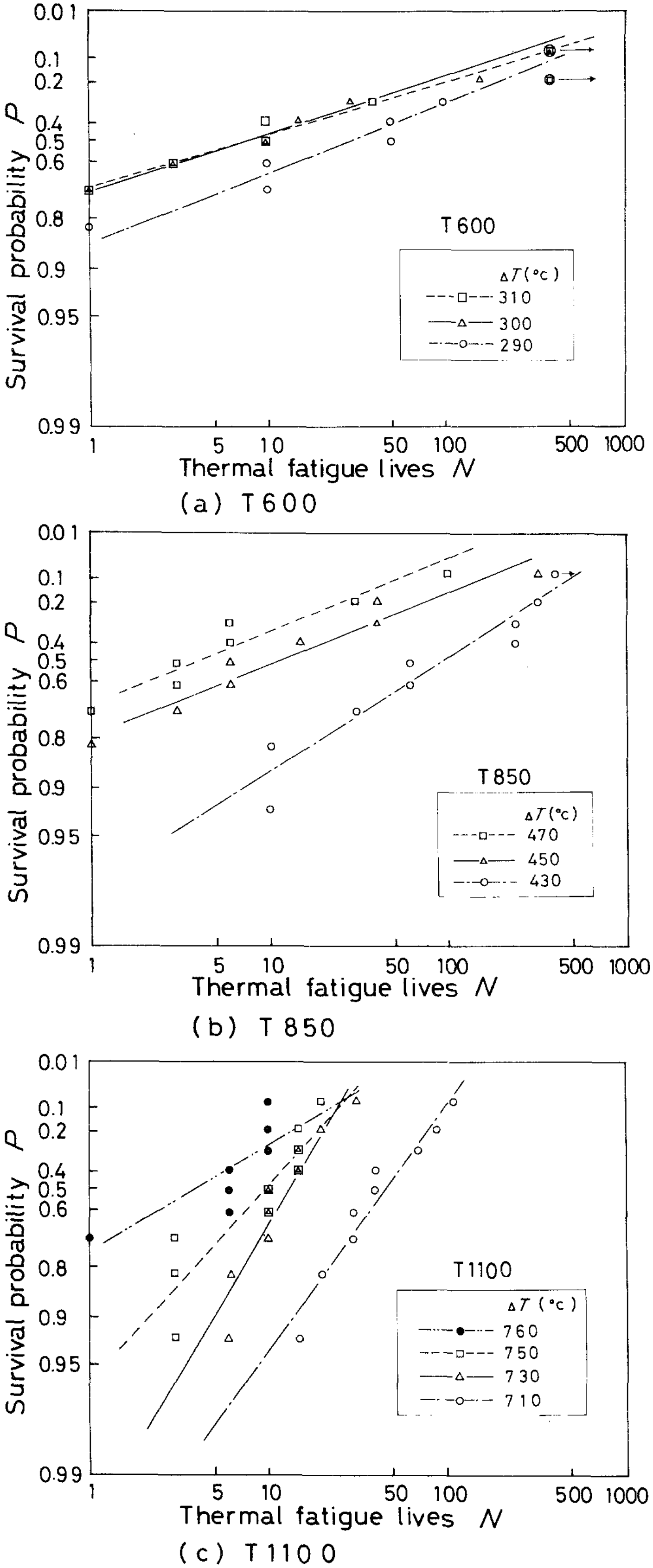

Figure 2 T-SPT diagram for glass ceramics at various stages of devitrification. Parameters in each figure indicate thermal shock severity, ΔT. Linear lines are obtained by linear regression analysis.

density of this glass is $0.195 \pm 0.05\,g\,cm^{-3}$ and its pore size is about 1 mm in average.

Thermal fatigue test was made by water quenching, which was described in a previous paper [2]. For this purpose, nine specimens held on a stage were heated in a furnace of a finely controlled temperature and they were rapidly transferred into water of a constant temperature, 30° C. The ends of each specimen were protected with tubes yarned with glass fibre to prevent them from the initiation of failure. The life of the ceramics was determined by the occurrence of fracture. The maximum number of repeated thermal shocks given to the specimens was limited to 400 cycles in this experiment.

The fracture stress of glass ceramics was also measured by 3-point bending with a crosshead speed of $0.5\,mm\,min^{-1}$ and a span of 30 mm in length.

Foamed glass was also used for the examination of the dependence of fracture stress on the stressing rate. The fracture stress was measured by 3-point bending with a crosshead speed from 0.05 to $5\,mm\,min^{-1}$, and a span was 80 mm in length.

The cyclic fatigue life of foamed glass was also determined by 3-point bending in a controlled atmosphere (relative humidity: 60%, temperature: $23 \pm 2°$ C). In bending tests, the span was 80 mm. The stress was applied by the way of a triangular function, whose cyclic period is 6 sec. The peak stress was varied from 0.106 to $0.126\,kg\,mm^{-2}$. In the test as well as the fracture stress measurement, asphalt impregnated felts of 0.5 mm in thickness were placed between the specimen and loading points to prevent stress concentration at the contact point.

The cumulative survival probability, P, for the stress of σ_i or the life of L_i was calculated as follows: first all data are arranged in the order of the value of fracture stress or fatigue life. The number, i (rank) is given for the datum which is at the i-th order in the data arranged as explained above. And the survival probability, P, for the stress, σ_i or the life, L_i, was calculated by the following equation,

$$P = 1 - \frac{i - 0.3}{J + 0.4} \tag{18}$$

where J stands for the number of the samples.

4. Results and discussion

The fracture stress of glass ceramics increased with the temperature of devitrification. For example, the averaged fracture stress is 18.4, 19.4 and $28.4\,kg\,mm^{-2}$, respectively, for T600, T850 and T1100 (Table II).

TABLE II Phases detected by the X-ray diffraction analysis in glass ceramics at various stages of devitrification

Heat treatment	Phase
T600: 600° C × 2 h (annealing)	Glass
T850: 850° C × 2 h	β-Spodumen, Glass
T1100: 850° C × 2 h → 1100° C × 2 h	β-Spodumen, Glass

Thermal fatigue life–survival probability curves (T-SPT diagram) for glass ceramics at different stages of devitrification are given in Fig. 2. The X-ray diffraction analysis reveals that the specimens consists of a crystalline phase as the major phase and glass as the minor one. The size of the dispersed crystalline phase was about 0.1 ~ 0.7 μm which is much smaller than a usual flaw size (30 ~ 100 μm). The phases detected are listed in Table II. According to the previous discussion, the crack growth in the specimens is mainly controlled by the crystalline phase. As shown in Figs. 2b and c, the specimen containing crystalline phase (T850, T1100) behaves in a different way from that of glass. The value of n estimated from the diagram is about 25 and 29 for the specimens heated at T850 and T1100, respectively. These values are a little different from that of glass, 20.

The T-SPT diagrams for glass ceramics are composed of lines running parallel except for the glass ceramics heated at T1100. The parallelism of the lines proves the presence of a major phase controlling the crack growth as well as the validity of the analysis. The T-SPT diagram for the glass ceramics heated at T1100 shows that the inclination of lines varies gradually with thermal shock severity (Fig. 2c). Similar variation in the inclination has been shown in the T-SPT diagram for sintered mullite [2], which is composed of a glassy phase and a crystalline phase, too. The variation in the inclination is thought to reflect the feature of a heterogeneous system as discussed previously.

A clear criterion for the change from the parallel lines in the T-SPT diagram (Fig. 2b) to unparallel lines in the diagram (Fig. 2c) is not known in this study.

The fracture stress distribution of foamed glass is shown for various crosshead speeds in Fig. 3. Weibull parameters determined from the figure are 17.9, 16.0, 14.8 and 19.8 for crosshead speeds

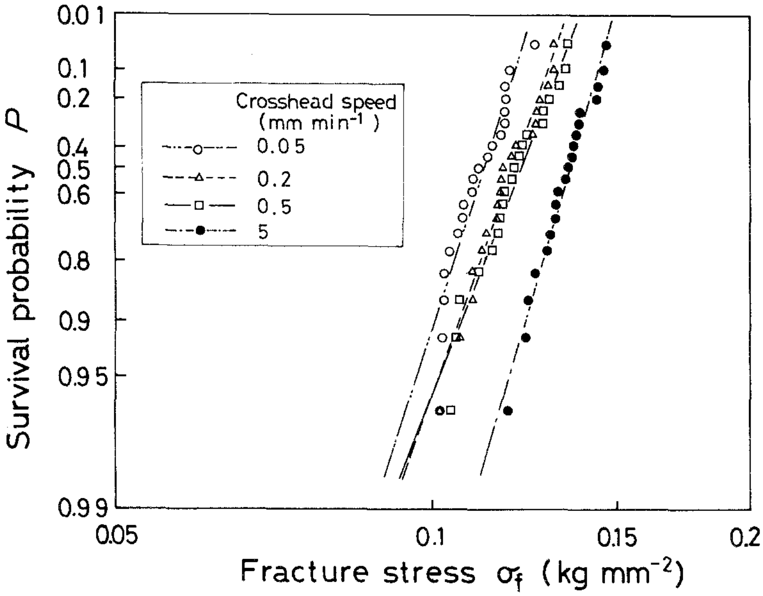

Figure 3 Fracture stress distribution of foamed glass for various crosshead speeds. Linear lines are obtained by linear regression analysis.

of 0.05, 0.2, 0.5 and 5 mm min^{-1}, respectively. Fracture stress for $P = 0.5$ are 0.111, 0.119, 0.121 and 0.135 kg mm^{-2}, respectively. The dependence of the fracture stress for $P = 0.5$ on crosshead speed is shown in Fig. 4. As shown in the figure, ln σ_f is proportional to ln (crosshead speed). This result agrees well with the conclusion mentioned previously. The value of n determined by applying Equation 17 to the figure is about 24. This value agrees well with the literature value [6] of glass. These results are thought to prove the validity of the formulae, also.

The survival probability–cyclic fatigue life for foamed glass is given in Fig. 5 with the peak stress as a parameter. The linear lines in the figure were obtained by the linear regression analysis. The lines for various peak stresses run parallel to one another. The parallelism agrees well with the result of the present approximated analysis. Weibull parameters for various peak stresses and the n value determined from Fig. 5 using Equation 15 are given in Table III. Weibull parameters determined from the cyclic fatigue test agree well with those from the fracture stress test (17.1 on average). Moreover, the value of n determined from the cyclic fatigue test agrees rather well with that determined from the dependence of fracture stress on stressing rate. These results and the parallelism in the cyclic fatigue test, are thought to prove the validity of the formulae given in the present study.

As shown above, the fatigue life as well as the fracture stress in a heterogeneous system can be dealt with in a similar way to that in a homogeneous system in some cases. But according to

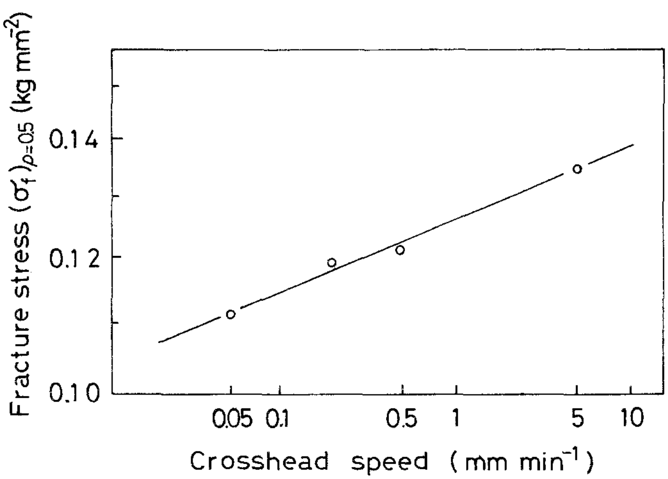

Figure 4 Dependence of fracture stress on crosshead speed for foamed glass.

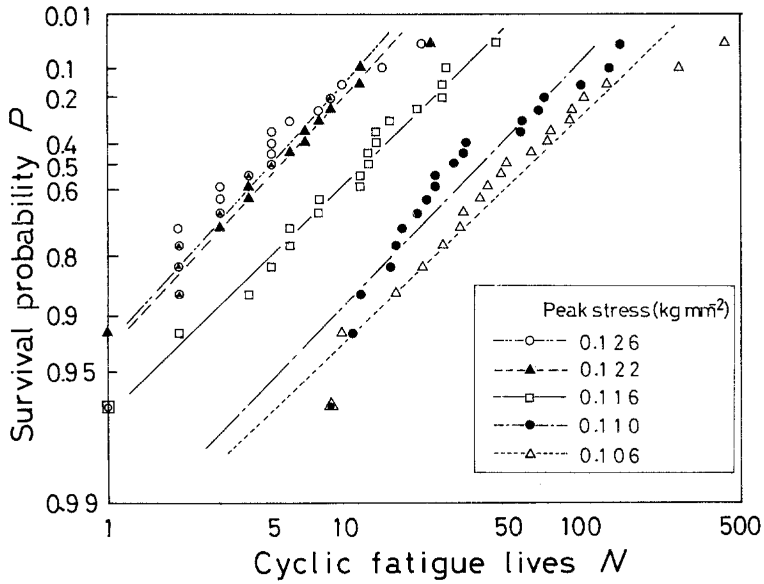

Figure 5 Distribution of cyclic fatigue life for foamed glass at various peak stresses. Linear lines are obtained by linear regression analysis.

the formulae derived in the present work, it can not be done when crack length is of the same order with the dimensions of the grain size, and when a ceramic is composed of many phases of different crack propagation speeds and when the concentration is of the same order. In such cases, a more tedious analysis including some numerical calculation for Equation 1 would be needed.

TABLE III Weibull parameters and n value determined from cyclic fatigue test for foamed glass

Peak stress (kg mm^{-2})	Weibull modulus, m	n
0.126	22.4	
0.122	21.5	
0.116	19.7	15.8
0.110	20.6	
0.106	18.0	

5. Conclusions

1. Some formulae for the prediction of fatigue life of ceramics of a heterogeneous system are given on the basis of slow crack growth.

2. The formulae are proved to be valid by the experiments on glass ceramics and foamed glass.

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Received 16 September 1982 and accepted 9 May 1983